

E-cospectral hypergraphs and some hypergraphs determined by their spectra

Changjiang Bu^a, Jiang Zhou^{a,b}, Yimin Wei^c

^aCollege of Science, Harbin Engineering University, Harbin 150001, PR China

^bCollege of Computer Science and Technology, Harbin Engineering University, Harbin 150001, PR China

^cSchool of Mathematical Sciences and Shanghai Key Laboratory of Contemporary Applied Mathematics, Fudan University, Shanghai, 200433, PR China

Abstract

Two k -uniform hypergraphs are said to be cospectral (E-cospectral), if their adjacency tensors have the same characteristic polynomial (E-characteristic polynomial). A k -uniform hypergraph H is said to be determined by its spectrum, if there is no other non-isomorphic k -uniform hypergraph cospectral with H . In this note, we give a method for constructing E-cospectral hypergraphs, which is similar with Godsil-McKay switching. Some hypergraphs are shown to be determined by their spectra.

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1. Introduction

For a positive integer n , let $[n] = \{1, \dots, n\}$. An order k dimension n tensor $\mathcal{A} = (a_{i_1 \dots i_k}) \in \mathbb{C}^{n \times \dots \times n}$ is a multidimensional array with n^k entries, where $i_j \in [n]$, $j = 1, \dots, k$. \mathcal{A} is called *symmetric* if $a_{i_1 i_2 \dots i_k} = a_{i_{\sigma(1)} i_{\sigma(2)} \dots i_{\sigma(k)}}$ for any permutation σ on $[k]$. We sometimes write $a_{i_1 \dots i_k}$ as $a_{i_1 \alpha}$, where $\alpha = i_2 \dots i_k$. When $k = 1$, \mathcal{A} is a column vector of dimension n . When $k = 2$, \mathcal{A} is an $n \times n$ matrix. The *unit tensor* of order $k \geq 2$ and dimension n is the tensor $\mathcal{I}_n = (\delta_{i_1 i_2 \dots i_k})$ such that $\delta_{i_1 i_2 \dots i_k} = 1$ if $i_1 = i_2 = \dots = i_k$, and

Email addresses: buchangjiang@hrbeu.edu.cn (Changjiang Bu),
zhoujiang04113112@163.com (Jiang Zhou), ymwei@fudan.edu.cn (Yimin Wei)

$\delta_{i_1 i_2 \dots i_k} = 0$ otherwise. When $k = 2$, \mathcal{I}_n is the identity matrix I_n . Recently, Shao [17] introduce the following product of tensors, which is a generalization of the matrix multiplication.

Definition 1.1. [17] *Let \mathcal{A} and \mathcal{B} be order $m \geq 2$ and order $k \geq 1$, dimension n tensors, respectively. The product \mathcal{AB} is the following tensor \mathcal{C} of order $(m-1)(k-1)+1$ and dimension n with entries:*

$$c_{i\alpha_1 \dots \alpha_{m-1}} = \sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} b_{i_2 \alpha_1} \dots b_{i_m \alpha_{m-1}},$$

where $i \in [n]$, $\alpha_1, \dots, \alpha_{m-1} \in [n]^{k-1}$.

Let \mathcal{A} be an order $m \geq 2$ dimension n tensor, and let $x = (x_1, \dots, x_n)^\top$. From Definition 1.1, the product $\mathcal{A}x$ is a vector in \mathbb{C}^n whose i -th component is (see Example 1.1 in [17])

$$(\mathcal{A}x)_i = \sum_{i_2, \dots, i_m \in [n]} a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m}.$$

In 2005, the concept of tensor eigenvalues was posed by Qi [13] and Lim [11]. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of \mathcal{A} , if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $\mathcal{A}x = \lambda x^{[m-1]}$, where $x^{[m-1]} = (x_1^{m-1}, \dots, x_n^{m-1})^\top$. The *determinant* of \mathcal{A} , denoted by $\det(\mathcal{A})$, is the resultant of the system of polynomials $f_i(x_1, \dots, x_n) = (\mathcal{A}x)_i$ ($i = 1, \dots, n$). The *characteristic polynomial* of \mathcal{A} is defined as $\Phi_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{I}_n - \mathcal{A})$, where \mathcal{I}_n is the unit tensor of order m and dimension n . It is known that eigenvalues of \mathcal{A} are exactly roots of $\Phi_{\mathcal{A}}(\lambda)$ (see [17]).

For an order $m \geq 2$ dimension n tensor \mathcal{A} , a number $\lambda \in \mathbb{C}$ is called an *E-eigenvalue* of \mathcal{A} , if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $\mathcal{A}x = \lambda x$ and $x^\top x = 1$. In [14], the *E-characteristic polynomial* of \mathcal{A} is defined as

$$\phi_{\mathcal{A}}(\lambda) = \begin{cases} \text{Res}_x \left(\mathcal{A}x - \lambda (x^\top x)^{\frac{m-2}{2}} x \right) & m \text{ is even,} \\ \text{Res}_{x, \beta} \begin{pmatrix} \mathcal{A}x - \lambda \beta^{m-2} x \\ x^\top x - \beta^2 \end{pmatrix} & m \text{ is odd,} \end{cases}$$

where ‘Res’ is the resultant of the system of polynomials. It is known that E-eigenvalues of \mathcal{A} are roots of $\phi_{\mathcal{A}}(\lambda)$ (see [14]). If $m = 2$, then $\phi_{\mathcal{A}}(\lambda) = \Phi_{\mathcal{A}}(\lambda)$ is just the characteristic polynomial of the square matrix \mathcal{A} .

A hypergraph H is called k -uniform if each edge of H contains exactly k distinct vertices. All hypergraphs in this note are uniform and simple. Let K_n^k denote the complete k -uniform hypergraph with n vertices, i.e., every k distinct vertices of K_n^k forms an edge. For a k -uniform hypergraph $H = (V(H), E(H))$, a hypergraph $G = (V(G), E(G))$ is a *sub-hypergraph* of H , if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. For any edge $u_1 \cdots u_k \in E(H)$, we say that u_k is a *neighbor* of $\{u_1, \dots, u_{k-1}\}$. The *complement* of H is a k -uniform hypergraph with vertex set $V(H)$ and edge set $E(K_{|V(H)|}^k) \setminus E(H)$. The *adjacency tensor* of H , denoted by \mathcal{A}_H , is an order k dimension $|V(H)|$ tensor with entries (see [2])

$$a_{i_1 i_2 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \text{if } i_1 i_2 \cdots i_k \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly \mathcal{A}_H is a symmetric tensor. We say that two k -uniform hypergraphs are *cospectral* (*E-cospectral*), if their adjacency tensors have the same characteristic polynomial (E-characteristic polynomial). A k -uniform hypergraph H is said to be *determined by its spectrum*, if there is no other non-isomorphic k -uniform hypergraph cospectral with H . We shall use “DS” as an abbreviation for “determined by its spectrum” in this note. Cospectral (E-cospectral) hypergraphs and DS hypergraphs are generalizations of cospectral graphs and DS graphs in the classic sense [3].

Recently, the research on spectral theory of hypergraphs has attracted extensive attention [2,6-9,12,15-20]. In this note, we give a method for constructing E-cospectral hypergraphs. Some hypergraphs are shown to be DS.

2. Preliminaries

The following lemma can be obtained from equation (2.1) in [17].

Lemma 2.1. *Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an order $m \geq 2$ dimension n tensor, and let $P = (p_{ij})$ be an $n \times n$ matrix. Then*

$$(P\mathcal{A}P^\top)_{i_1 \cdots i_m} = \sum_{j_1, \dots, j_m \in [n]} a_{j_1 \cdots j_m} p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_m j_m}.$$

We can obtain the following lemma from Lemma 2.1.

Lemma 2.2. *Let $\mathcal{B} = P\mathcal{A}P^\top$, where \mathcal{A} is a tensor of dimension n , P is an $n \times n$ matrix. If \mathcal{A} is symmetric, then \mathcal{B} is symmetric.*

Let $\mathcal{B} = P\mathcal{A}P^\top$, where \mathcal{A} is a tensor of dimension n , P is an $n \times n$ real orthogonal matrix. In [17], Shao pointed out that \mathcal{A}, \mathcal{B} are orthogonally similar tensors defined by Qi [13]. Orthogonally similar tensors have the following property.

Lemma 2.3. [10] *Let $\mathcal{B} = P\mathcal{A}P^\top$, where \mathcal{A} is a tensor of dimension n , P is an $n \times n$ real orthogonal matrix. Then \mathcal{A} and \mathcal{B} have the same E -characteristic polynomial.*

A *simplex* in a k -uniform hypergraph is a set of $k+1$ vertices where every set of k vertices forms an edge (see [2, Definition 3.4]).

Lemma 2.4. *Let G and H be cospectral k -uniform hypergraphs. Then G and H have the same number of vertices, edges and simplices.*

Proof. The degree of the characteristic polynomial of an order k dimension n tensors is $n(k-1)^{n-1}$ (see [13]). Since \mathcal{A}_G and \mathcal{A}_H are order k tensors, G and H have the same number of vertices. From [2, Theorem 3.15] and [2, Theorem 3.17], we know that G and H have the same number of edges and simplices. \square

3. Main results

Let $H = (V(H), E(H))$ be a k -uniform hypergraph with a partition $V(H) = V_1 \cup V_2$, and H satisfies the following conditions:

- (a) For each edge $e \in E(H)$, e contains at most one vertex in V_1 .
- (b) For any $k-1$ distinct vertices $u_1, \dots, u_{k-1} \in V_2$, $\{u_1, \dots, u_{k-1}\}$ has either $0, \frac{1}{2}|V_1|$ or $|V_1|$ neighbors in V_1 .

Similar with GM switching [4, 5], we construct a hypergraph E -cospectral with H as follows.

Theorem 3.1. *Let H be a k -uniform hypergraph satisfies the conditions (a) and (b) described above. For any $\{u_1, \dots, u_{k-1}\} \subseteq V_2$ which has $\frac{1}{2}|V_1|$ neighbors in V_1 , by replacing these $\frac{1}{2}|V_1|$ neighbors with the other $\frac{1}{2}|V_1|$ vertices in V_1 , we obtain a k -uniform hypergraph G which is E -cospectral with H .*

Proof. Let $P = \begin{pmatrix} \frac{2}{n_1}J - I_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix}$, where $n_1 = |V_1|, n_2 = |V_2|$, J is the $n_1 \times n_1$ all-ones matrix, $\frac{2}{n_1}J - I_{n_1}$ and I_{n_2} correspond to the vertex sets V_1 and V_2 , respectively. Then $P = P^\top = P^{-1}$. Suppose that $\mathcal{A}_H = (a_{i_1 i_2 \dots i_k})$, and

let $\mathcal{B} = P\mathcal{A}_H P^\top$. By Lemma 2.2, \mathcal{B} is symmetric. We need to show that $\mathcal{B} = \mathcal{A}_G$. By Lemma 2.1, we have

$$(\mathcal{B})_{i_1 \dots i_k} = \sum_{j_1, \dots, j_k \in V(H)} a_{j_1 \dots j_k} p_{i_1 j_1} p_{i_2 j_2} \dots p_{i_k j_k}. \quad (1)$$

Note that $P = \begin{pmatrix} \frac{2}{n_1}J - I_{n_1} & 0 \\ 0 & I_{n_2} \end{pmatrix}$. From Eq. (1), we have

$$(\mathcal{B})_{i_1 \dots i_k} = a_{i_1 \dots i_k} \text{ if } i_1, \dots, i_k \in V_2. \quad (2)$$

Since H satisfies the condition (a), we have $a_{j_1 \dots j_k} = 0$ if $|\{j_1, \dots, j_k\} \cap V_1| \geq 2$. From Eq. (1), we have

$$(\mathcal{B})_{i_1 \dots i_k} = a_{i_1 \dots i_k} = 0 \text{ if } |\{i_1, \dots, i_k\} \cap V_1| \geq 2. \quad (3)$$

Next we consider the case $|\{i_1, \dots, i_k\} \cap V_1| = 1$. Note that \mathcal{B} is symmetric. Without loss of generality, suppose that $i_1 \in V_1, i_2, \dots, i_k \in V_2$. From Eq. (1), we have

$$(\mathcal{B})_{i_1 \dots i_k} = \sum_{j_1 \in V_1} a_{j_1 i_2 \dots i_k} p_{i_1 j_1} \text{ (} i_1 \in V_1, i_2, \dots, i_k \in V_2 \text{)}. \quad (4)$$

Since H satisfies the condition (b), $S^{i_2 \dots i_k} = \{a_{j_1 i_2 \dots i_k} | j_1 \in V_1, a_{j_1 i_2 \dots i_k} \neq 0\}$ contains either $0, \frac{1}{2}|V_1|$ or $|V_1|$ elements for any given $i_2, \dots, i_k \in V_2$. By computing the sum in (4), we have

$$(\mathcal{B})_{i_1 \dots i_k} = a_{i_1 \dots i_k} = 0 \text{ if } i_1 \in V_1, i_2, \dots, i_k \in V_2, |S^{i_2 \dots i_k}| = 0. \quad (5)$$

$$(\mathcal{B})_{i_1 \dots i_k} = a_{i_1 \dots i_k} = \frac{1}{(k-1)!} \text{ if } i_1 \in V_1, i_2, \dots, i_k \in V_2, |S^{i_2 \dots i_k}| = |V_1|. \quad (6)$$

$$(\mathcal{B})_{i_1 \dots i_k} = 0 \text{ if } i_1 \in V_1, i_2, \dots, i_k \in V_2, a_{i_1 \dots i_k} = \frac{1}{(k-1)!}, |S^{i_2 \dots i_k}| = \frac{1}{2}|V_1|. \quad (7)$$

$$(\mathcal{B})_{i_1 \dots i_k} = \frac{1}{(k-1)!} \text{ if } i_1 \in V_1, i_2, \dots, i_k \in V_2, a_{i_1 \dots i_k} = 0, |S^{i_2 \dots i_k}| = \frac{1}{2}|V_1|. \quad (8)$$

From Eqs. (2)(3) and (5)-(8), we have $\mathcal{B} = P\mathcal{A}_H P^\top = \mathcal{A}_G$. By Lemma 2.3, G is E-cospectral with H . \square

If two k -uniform hypergraphs G and H are isomorphic, then there exists a permutation matrix P such that $\mathcal{A}_G = P\mathcal{A}_H P^\top$ (see [1, 17]). From Lemma 2.3, we know that two isomorphic k -uniform hypergraphs are E-cospectral. By using the method in Theorem 3.1, we give a class of non-isomorphic E-cospectral hypergraphs as follows.

Example. Let H be a 3-uniform hypergraph whose vertex set and edge set are

$$\begin{aligned} V(H) &= \{u_1, u_2, u_3, u_4, v_1, \dots, v_n\} \ (n \geq 3), \\ E(H) &= \{v_1 v_2 u_2, v_1 v_2 u_3, v_2 v_3 u_2, v_2 v_3 u_4, v_1 v_3 u_3, v_1 v_3 u_4\} \cup F, \end{aligned}$$

where each edge in F contains three vertices in $\{v_1, \dots, v_n\}$, and each vertex in $\{v_4, \dots, v_n\}$ is contained in at least one edge in F if $n \geq 4$. Let G be a 3-uniform hypergraph whose vertex set and edge set are

$$V(G) = V(H), E(G) = \{v_1 v_2 u_1, v_1 v_2 u_4, v_2 v_3 u_1, v_2 v_3 u_3, v_1 v_3 u_1, v_1 v_3 u_2\} \cup F.$$

The vertex set $V(H)$ has a partition $V(H) = V_1 \cup V_2$ such that H and G satisfy the conditions in Theorem 3.1, where $V_1 = \{u_1, u_2, u_3, u_4\}$, $V_2 = \{v_1, \dots, v_n\}$. Then G and H are E-cospectral. Moreover, G and H are non-isomorphic E-cospectral hypergraphs, because H has an isolated vertex u_1 and G has no isolated vertices.

Let $K_n^k - e$ denote the k -uniform hypergraph obtained from K_n^k by deleting one edge. We can obtain the following result from Lemma 2.4.

Theorem 3.2. *The complete k -uniform hypergraph K_n^k , the hypergraph $K_n^k - e$ and their complements are DS. Any k -uniform sub-hypergraph of K_{k+1}^k is DS. The disjoint union of K_{k+1}^k and some isolated vertices is DS.*

If G is a sub-hypergraph of a hypergraph H , then let $H \setminus G$ denote the hypergraph obtained from H by deleting all edges of G .

Theorem 3.3. *The hypergraph $K_n^k \setminus G$ is DS, where G is a k -uniform sub-hypergraph of K_n^k such that all edges of G share $k - 1$ common vertices.*

Proof. Let H be any k -uniform hypergraph cospectral with $K_n^k \setminus G$. Suppose that G has r edges. By Lemma 2.4, H can be obtained from K_n^k by deleting r edges. Deleting r edges from K_n^k destroys at least $\sum_{i=0}^{r-1} (n - k - i)$ simplices, with equality if and only if all deleted edges share $k - 1$ common vertices. Lemma 2.4 implies that $H = K_n^k \setminus G$. \square

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